Bounds on Certified Domination number in Graphs

Bachha Nagesh [1] Kishori P. Narayankar [2]

[1]Vani vidyalaya jr college J.N Road Mulund West Mumbai, India- 400080
[2] Department of Mathematics, Mangalore University, Mangalore-574199, India.

{kishori_pn@yahoo.co.in,}

May 7, 2025

Abstract

A dominating set D of a graph $G = (V, E)$ is said to be certified if every vertex in D has either zero or at least two neighbours in $V - D$. The cardinality of minimum certified dominating set in G is called the certified domination number of G. It is denoted by $\gamma_{cer}(G)$ In this paper we give a bounds for certified domination number and also show that every graph G is an induced subgraph of some super graph H such that certified domination number $\gamma_{cer}(G) = 2$

1 Introduction

By a graph $G = (V(G), E(G))$ we mean a finite, undirected connected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by $nandm$, respectively. For graph theoretic terminology we refer to [].
The *degree* $d(v)$of$v$ is $|N(v)|$. If $d(v) = 1$, then a vertex v is called a *pendant vertex* and the unique vertex u which is adjacent to v is called a *support vertex*. A support vertex u is called a *strong support vertex* if the number of pendant vertices adjacent to a vertex u is at least two. Otherwise u is called a *weak support vertex*. Then the induced subgraph $G[S]$ is the graph whose vertex set is S and edge set consists of all of the edges in $E(G)$ that have both endpoints in S.

let S be a subset of the vertex set of a graph $G = (V(G), E(G))$. We say that S dominates G (or is a dominating set of G) if each vertex in the set $V(G) - S$ has a neighbour in S. The cardinality of a minimum dominating set in G is called the domination number of G and denoted by $\gamma(G)$, and any minimum dominating set of G is called a γ-set. A dominating set S of G is called certified if every vertex $v \in D$has either zero or at least two neighbours in $V(G) - S$. The cardinality of a minimum certified dominating set in G is called the certified domination number of G and denoted by $\gamma_{cer}(G)$. A minimum certified dominating set of G is called a γ_{cer}-set. One can observe that, by the definition, $V(G)$ is a certified dominating set of G.

2 Bounds on Certified Domination Number

It is observed and studied in [1] that every support vertex of a graph G belongs to every certified dominating set S of G. And hence if the strong supports of G are adjacent to k

leaves in total then

$$\gamma_{cer}(G) \leq n - k.$$

One can observe that $\gamma_{cer} - set \neq \emptyset$. And so, $|S| \geq 1$. Clearly,

$$1 \leq \gamma_{cer}(G) \leq n.$$

Lower bound equality holds if $\Delta(G) = n - 1$ and the upper bound equality holds if G is isomorphic to K_n and $n = 2$. The bound of Ore [] also holds good for certified domination number $\gamma_{cer}(G)$. That is,

$$\gamma_{cer}(G) \leq \frac{n}{2}.$$

For instance one can refer the following result:

Theorem 2.1. *[] If G is a connected graph with $\delta(G) \geq 2$, then $\gamma_{cer}(G) \leq n/2$.*

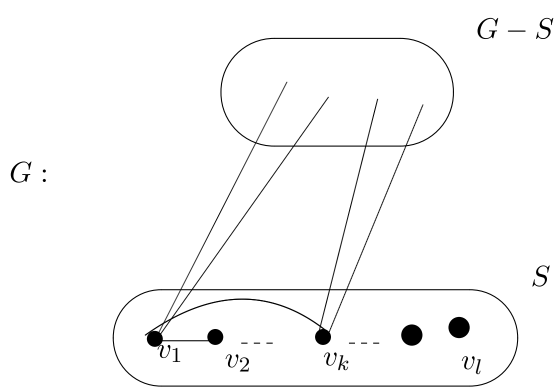
Proposition 2.2. *Let G be any connected graph of order n and size m with $\delta(G) \geq 2$. Let S be certified dominating set. Then $V - S$ is also certified dominating set.*

Proof. Let G be a graph of order n, size m and $\delta(G) \geq 2$. Let S be any CDS. We claim that $V - S$ is CDS. Since $\delta(G) \geq 2$ every vertex of $V - S$ has at least two neighbours in S. Hence $V - S$ is CDS. □

Theorem 2.3. *Let G be any connected graph of order n and size m with $\delta(G) \geq 2$. Then*

$$\gamma_{cer}(G) \leq n - \delta(G) + k.$$

Proof. Let S be CDS in G and let $S = \{v_1, v_2, v_3, \ldots, v_l\}$ Assume that $v_1 \in S$ is adjacent to k vertices in S. Then every vertex of S has either zero neighbors or has at least two neigbors in $V - S$. Clearly, if $v_1 \in S$ is adjacent to k vertices in S then it has atleast two neighbour in $V - S$.



Since $degv_1 = \delta(G) \geq 2$ (at least), v_1 must be adjacent to atleast $\delta(G) - k$ vertices in $V - S$. If $k = 0$, then $|V - S| \geq \delta(G)$ and so $|S| \leq n - \delta(G)$. if $k \neq 0$ then each neighbour of v_1 in S is adjacent to atleast two vertices in $|V - S|$. And observe that these k vertices are all distict. Hence $|V - S| \geq \delta(G) - k + 2k = \delta(G) + k$. Therefore $|S| \leq n - \delta(G) + k$. □

Theorem 2.4. *Let G be a connected graph of order n and size m with $\delta(G) \geq 2$. Then*

$$\gamma_{cer}(G) \leq n - \Delta(G)$$

Proof. □

Theorem 2.5. *For any connected graph G, $\gamma_{cer}(G) \leq 2m - n$. Equality holds if and only if G is either P_3or P_4.*

Proof. It observed [] that there does not exist a graph for which the following bound holds:

$$\gamma_{cer} G \leq n - 1.$$

Hence

$$\begin{aligned}
\gamma_{cer}(G) &\leq n - 2 \\
&= n - 2 + n - n \\
&= 2(n-1) - n \qquad \text{Since } G \text{ is connected } m \geq n-1 \\
&= 2m - n.
\end{aligned}$$

If, $2m - n = n - 2$ then, $m = n - 1$. And so G is a tree. If k are number of supports in G then,

$$\gamma_{cer}(G) \leq n - k.$$

Clearly $k \geq 2$. If $k = 2$, then G is a path graph on n vertices.

$$\begin{aligned}
\gamma_{cer}(G) &\leq n - 2 \\
&= 2m - n \qquad \text{holds}
\end{aligned} \tag{1}$$

Let $k > 2$, then $\gamma_{cer}(G) = n - k < n - 2 = 2m - n$. A contradiction as $\gamma_{cer}(G) \leq 2m - n$. Hence $k \leq 2$, but since G is a tree with at least two supports, i.e., $k \geq 2$, $\implies k = 2$ and so G is a path graph on n vertices. Let G be a path graph of order $n \geq 5$ from [] ,$\gamma_{cer}(P_n) = [n/3]$ for $n \geq 5$. Hence for $k = 2$ $\gamma_{cer}(G) = n - 2$ $[n/3] = n - 2$ is not true in general for any n≥ 5. For if $n = 2$ in $[n/3] = n - 2$, the result is absord . Hence $n = 3$ and $n = 4$ and so, $G = P_3$ and $G = P_3$. □

Theorem 2.6. *Let $r \geq 2$ be any integer G be r-regular greph then $\gamma_{cer}(G) \leq n - r$. Equality holds for complete graph*

Proof. Let $r \geq 2$ be any integer. let $degv = r$ for any $v \in V(G)$. Then v dominates itself and $N[v]$, And the vertices in $V - N[V]$ forms certified dominating set, as every $v \in V - N[V]$ has atleast two neighbours. Hence $V - N[V]$ is a certified dominating set of cardinality $n - r$. Hence

$$\gamma_{cer}(G) \leq n - r$$

□

3 Characterization of graph for $\gamma_{cer}(G) = 2$ and $\delta(G) \geq 1$

One can observe that in a connected graph If $degv = 1$, for all $v \in V(G)$ then $\gamma_{cer}(G) = 2$ and $\delta(G) = 1$ is trivially true. In this section we focus on $\gamma_{cer}(G) = 2$ and $\delta(G) \geq 1$.

Theorem 3.1. *For any graph G of order n and size m, $\gamma_{cer}(G) = 2$ and $\delta(G) = 1$ if and only if G has atmost two supports u and v such that $d(u, v) \leq 3$.*

Proof. Let G be with atmost two supports u and v such that $d(u, v) \leq 3$. The following are the possible graphs H_i, $i = 1, 2, 3$: In P_4, a path on four vertices, replace a central vertex u by a graph G such that a resultant graph H_1 is a graph with one single support. Next, take two copies of K_2, a complete graph on two vertices and any graph G.To one vertex of K_2 join all vertices of G and to all vertices of G join another vartex of another copy of K_2 such that a resultant graph H_2 is a graph with two supports. Finally, take P_3, a path on three vertices and K_2, a complete graph on two vertices. join one vertex of $P_3$to all vertices of G and all vertices of G to one vertex of K_2 and that a resultant graph H_3 is a graph with two supports again. In all the cases with $d(u, v) \leq 3$ where u and v are atmost two supports we have $\gamma_{cer}(G) = 2$ with $\delta(G) \geq 2$.
Let $\gamma_{cer}(G) = 2$ with $\delta(G) \geq 2$. Suppose G is a graph with atmost two supports u and v and that $d(u, v) > 3$. Then G has $\gamma_{cer}(G) > 2$. □

Using Theorem 3.1 it is easy to check that every graph G is an induced subgraph of some super graph H such that certified domination number $\gamma_{cer}(G) = 2$ and $\delta(G) \geq 1$.

to characterise the class of critical graphs where the certified domination number increases on the removal of any edge/vertex as well as the class of stable graphs where the certified domination number remains unchanged on the removal of any edge/vertex with minimum degree $\delta(G) \leq 2$ is an open problem []. A small attempt is made to solve this open problem by restricting ourselves for $\gamma_{cer}(G) = 2$ and $\delta(G) = 1$.

Theorem 3.2. *Any graph G with $\gamma_{cer}(G) = 2$ and $\delta(G) = 1$. is critical graph if and only if $G = P_4$.*

Theorem 3.3. *Any tree T with $\gamma_{cer}(G) = rad(G) + 1$ are critical trees.*

References

[1] S. Durai Raj, S. G. Shiji Kumari and A. M. Anto Certified domination number in corona product of graphs *Malaya Journal of Matematik, Vol. 9, No. 1, 1080-1082*, 2021.

[2] R.C. Brigham, P.Z. Chinn, R.D. Dutton, Vertex domination-critical graphs, Networks 18 (1988) 173–179.

[3] D.B. West, Introduction to Graph Theory, Prentice hall, Upper Saddle River, 2001.

[4] J. Fulman, D. Hanson, G. MacGillivray, Vertex domination-critical graphs, Networks 25 (1995) 41–43.

[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.

[7] M.A. Henning, D.F. Rall, On graphs with disjoint dominating and 2-dominating sets, Discuss. Math. Graph Theory 33 (2013) 139–146.

[8] R. Diestel, Graph Theory, Springer, Heidelberg, 2012.

99